Nonlinearly-$\mathcal{PT}$-symmetric systems: spontaneous symmetry breaking and transmission resonances

Andrey E. Miroshnichenko[1], Boris A. Malomed[2], and Yuri S. Kivshar[1]

[1]*Nonlinear Physics Centre, Australian National University, Canberra ACT 0200, Australia*
[2]*Department of Physical Electronics, School of Electrical Engineering,*
Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel

We introduce a class of $\mathcal{PT}$-symmetric systems which include mutually matched *nonlinear* loss and gain (in other words, a class of $\mathcal{PT}$-invariant Hamiltonians in which both the harmonic and anharmonic parts are non-Hermitian). For a basic system in the form of a dimer, symmetric and asymmetric eigenstates, including multistable ones, are found analytically. We demonstrate that, if coupled to a linear chain, such a nonlinear $\mathcal{PT}$-symmetric dimer generates new types of nonlinear resonances, with the completely suppressed or greatly amplified transmission, as well as a regime similar to the electromagnetically-induced transparency (EIT). The implementation of the systems is possible in various media admitting controllable linear and nonlinear amplification of waves.

Introduction. In the past few years, the study of systems exhibiting the parity-time ($\mathcal{PT}$) symmetry has drawn a great deal of attention. The underlying idea is to extend canonical quantum mechanics by introducing a class of non-Hermitian Hamiltonians which may, nevertheless, exhibit entirely real eigenvalue spectra [1]. A necessary condition for the Hamiltonian to be $\mathcal{PT}$-symmetric is that its linear-potential part $V(x)$, being complex, is subject to the spatial-symmetry constraint, $V(x) = V^*(-x)$. The complex $\mathcal{PT}$-symmetric potentials can be realized in the most straightforward way in optics, by combining the spatial modulation of the refractive index with properly placed gain and loss [2]. This possibility has stimulated extensive theoretical [3] and experimental [4] studies.

In the $\mathcal{PT}$-symmetric Hamiltonians introduced in the context of the field theory and optics, solely the harmonic part features the matched gain and loss, while the anharmonic part, if any, is Hermitian, giving rise to nonlinear dynamical models in which only the linear part features the balanced dissipation and amplification [5]. In this work, we put forward an extension of the $\mathcal{PT}$ symmetry, in the form of Hamiltonians whose anharmonic part too includes the mutually matched loss and gain. Solving the corresponding dynamical equations, we demonstrate that such *nonlinearly-$\mathcal{PT}$-symmetric* systems also give rise to eigenstates with real frequencies. New findings, which are specific to the systems with the matched nonlinear gain and loss, are eigenstates with a spontaneously broken spatial symmetry (that, as mentioned above, implements the $\mathcal{PT}$ symmetry of the Hamiltonian), and multistability of eigenstates. A straightforward application of these states is realized by coupling the $\mathcal{PT}$ system to linear chains: we demonstrate that this setting gives rise to new types of multistable nonlinear Fano resonances, as well as to transmission regimes with a very strong amplification, and those similar to EIT (electromagnetically-induced transparency). Systems of this type can be im-

plemented in optics, using saturable absorbers [6] and two-photon losses to realize the nonlinear $\mathcal{PT}$ symmetry, as well as in any medium which allows nonlinear amplification of waves, such as cavity polaritons [7], surface plasmons [8], and magnons [9].

We start by introducing a solvable nonlinearly-$\mathcal{PT}$-symmetric system, in the form of a dimer, in which the symmetric linear gain and loss terms come along with their nonlinear mutually conjugate counterparts,

$$
\begin{aligned}
i\dot{\psi}_A &= (E + i\gamma_0 - i\gamma_2|\psi_A|^2 + \chi|\psi_A|^2)\psi_A + V\psi_B, \\
i\dot{\psi}_B &= (E - i\gamma_0 - i\gamma_2|\psi_A|^2 + \chi|\psi_B|^2)\psi_B + V\psi_A.
\end{aligned}
\tag{1}
$$

Here the overdot stands for the time derivative, $\gamma_0 > 0$ accounts for the linear gain and loss, which act on complex variables ψ_A and ψ_B, respectively, E is a frequency shift (with respect to a linear chain if the dimer is coupled to it, see below), γ_2 accounts for the $\mathcal{PT}$-symmetric nonlinear loss and gain (as shown below, stable eigenstates are obtained with $\gamma_2 > 0$, i.e., if the nonlinear loss competes with the linear gain, and vice versa), χ is the strength of the nonlinear frequency shift, and V is a coupling coefficient.

Symmetric modes. Symmetric eigenstates, with $|\psi_A| = |\psi_B|$, are sought for as $\psi_{A,B}(t) = A \exp(-i\omega t \pm i\delta/2)$, with the amplitude and phase shift determined by the following equations:

$$
\left[\chi A^2 - (\omega - E)\right]^2 + \left(\gamma_0 - \gamma_2 A^2\right)^2 = V^2, \tag{2}
$$

$$
\tan\delta = \left(\gamma_0 - \gamma_2 A^2\right)\left(\omega - E - \chi A^2\right)^{-1}. \tag{3}
$$

Depending on the parameters, Eq. (2) may yield no physical solutions with $A^2 > 0$, a single solution (monostability), and bistability, with two physical roots. The bistability occurs under conditions

$$
\begin{aligned}
(\omega - E)^2 &> V^2 - \gamma_0^2, \\
\gamma_0\gamma_2\chi^{-1} + (\omega - E) &> \sqrt{(\omega - E)^2 + \gamma_0^2 - V^2},
\end{aligned}
\tag{4}
$$

while the monostability condition is $(\omega - E)^2 < V^2 - \gamma_0^2$. Although the system is dissipative, the symmetric eigenstates with the real frequencies form a continuous family parameterized by arbitrary frequency ω, which is a manifestation of the $\mathcal{PT}$ symmetry.

Asymmetric states and multistability. The system admits solutions with broken symmetry too ($A \neq B$): $\psi_{A,B}(t) = \left\{ A e^{i\delta/2}, B e^{-i\delta/2} \right\} e^{-i\omega t}$, with δ determined by the same equation (3) as above. Unlike the symmetric eigenstates, the asymmetric ones exist at a single frequency, which is typical to dissipative systems:

$$\omega_{AS} = E + (\gamma_0/\gamma_2)\chi. \tag{5}$$

Further, amplitude A is determined by equation

$$\left(A^2\right)^2 - (\gamma_0/\gamma_2)A^2 + V^2\left(\chi^2 + \gamma_2^2\right)^{-1} = 0, \tag{6}$$

cf. Eq. (2), while the other amplitude is given by $B^2 = (\gamma_0/\gamma_2) - A^2$, which shows that the asymmetric eigenmode exists only for $\gamma_2 > 0$, i.e., when the nonlinear $\mathcal{PT}$-symmetric loss/gain terms compete with their linear counterparts. We stress that the asymmetric solutions are supported by the balance of the nonlinear gain and loss, as they do not exist for $\gamma_2 = 0$.

The above relations yield two physical solutions (i.e., the bistability) for the asymmetric modes, with $A^2, B^2 > 0$, at $\chi^2/\gamma_2^2 > 4\left(V^2/\gamma_0^2\right) - 1$, and no solutions in the opposite case. Under this condition, inequalities (4) hold too for $\omega = \omega_{AS}$, i.e., the system gives rise to the *multistability*, with four coexisting eigenstates, two symmetric and two asymmetric.

The scattering problem. The next step is to couple the dimer to a chain transmitting linear discrete waves $\psi_n(t)$, as shown in Fig. 1 (note that the entire system remains $\mathcal{PT}$-symmetric). Here we focus on the most fundamental version of the system, with $\chi = 0$, the nonlinearity being represented by the matched cubic loss and gain, the respective coupled system being

$$i\dot{\psi}_A = E\psi_A + i\left(\gamma_0 - \gamma_2|\psi_A|^2\right)\psi_A + V\psi_0, \tag{7}$$

$$i\dot{\psi}_n = C\left(\psi_{n-1} + \psi_{n+1}\right) + V\delta_{n,0}\left(\psi_A + \psi_B\right), \tag{8}$$

$$i\dot{\psi}_B = E\psi_B - i\left(\gamma_0 - \gamma_2|\psi_B|^2\right)\psi_B + V\psi_0, \tag{9}$$

where C is the coupling constant in the linear chain. The general solution corresponding to the scattering of incident waves with amplitude I on the $\mathcal{PT}$ complex is looked for as

$$\psi_n = \begin{cases} Ie^{i(kn - \omega t)} + Re^{-i(kn + \omega t)} & (n \leq 0), \\ Te^{i(kn - \omega t)} & (n \geq 0), \end{cases} \tag{10}$$

where wavenumber $k > 0$ is determined by the dispersion equation for the linear chain, $k = \cos^{-1}(\omega/2C)$, while R and T are the amplitudes of reflected and transmitted waves. A straightforward analysis of Eqs. (8,10) at $n = 0$
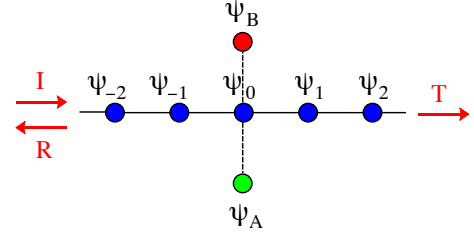

FIG. 1. (Color online) The linear chain with the side-coupled elements featuring the nonlinear $\mathcal{PT}$ symmetry. The arrows indicate incident, reflected and transmitted waves.

yields $R = \psi_0 - I$, $T = \psi_0$, and the expression for ψ_0 in terms I and $\psi_{A,B}^{(0)}$:

$$\psi_0 = I + iV\left(2C\sin k\right)^{-1}\left(\psi_A^{(0)} + \psi_B^{(0)}\right). \tag{11}$$

The substitution of expression (11) into the stationary version of Eqs. (7) and (9) leads to a system of complex cubic equations:

$$(E - \omega)\psi_{A,B}^{(0)} + iV^2\left(2C\sin k\right)^{-1}\left(\psi_A^{(0)} + \psi_B^{(0)}\right)$$
$$\pm i\left(\gamma_0 - \gamma_2\left|\psi_{A,B}^{(0)}\right|^2\right)\psi_{A,B}^{(0)} = -VI. \tag{12}$$

One should solve Eq. (12) for $\psi_{A,B}^{(0)}$ at given I and ω. Then, ψ_0 can be found from Eq. (11), and, eventually, the reflection and transmission coefficients can be found.

The scattering in the symmetric regime. In the linear system ($\gamma_2 = 0$), Eq. (12) yields only symmetric solutions, with $|\psi_A| = |\psi_B|$. The corresponding scattering spectrum, displayed in Fig. 2, demonstrates two noteworthy effects. One is the suppression of the transmission by the degenerate side-coupled elements without the gain and loss, $\gamma_0 = 0$. In this case, the eigenfrequencies of both elements are identical, and their excitation results in the resonant reflection at $\omega = E$, which can be explained in terms of the Fano resonance [10]. The presence of the weak linear gain and loss, with $\gamma_0 \ll 1$, lifts the degeneracy between the attached sites, leading to a response resembling the electromagnetically-induced transparency (EIT) [11], with the total transmissivity ($T = 1$) at $\omega = E$, between resonant reflections on the pair of slightly detuned linear $\mathcal{PT}$ elements.

In the system combining the linear chain and the nonlinear $\mathcal{PT}$ scatterer with $\gamma_2 > 0$, one can find symmetric solutions with $\psi_A^{(0)} = -\psi_B^{(0)} = -i\phi$, where ϕ is real. First, we consider the case of $\phi \neq \sqrt{\gamma_0/\gamma_2}$ (this value plays a special value, as shown below). Then, the symmetric mode exists at $\omega = E$, with ϕ determined by equation
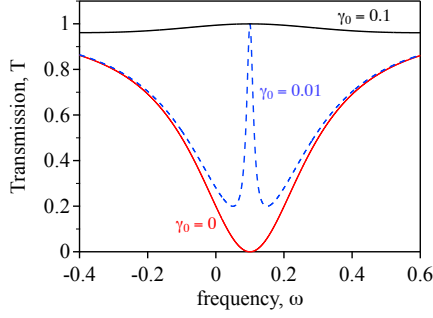
$$\gamma_2\phi^3 - \gamma_0\phi + VI = 0, \tag{13}$$

FIG. 2. (Color online) The transmission coefficient for several values of the gain/loss factor γ_0 in the linear system ($\gamma_2 = 0$). Other parameters are $E = 0.1$, $V = 0.2$, and $C = 1$.

which yields a single real root for $P_{\text{in}} \equiv I^2 > (4/27)\,\gamma_0^3/\left(V^2\gamma_2\right)$, and three real solutions (*tristability*) in the opposite case. According to Eq. (11), all these solutions realize the perfect EIT-like transmissivity, with $T \equiv 1$ [the horizontal blue line in Fig. 3(a)]. The family of the symmetric states is displayed by the blue curves in Fig. 3(b), where the tristability occurs at $P_{\text{in}} < 1/27$.

Nonlinear Fano resonances. In contrast to its linear counterpart, the nonlinear system may support complete suppression of the transmission ($T = 0$), i.e., nonlinear Fano resonances [10]. From Eq. (11) it follows that $\psi_A^{(0)} + \psi_B^{(0)} = 2iICV^{-1}\sin k$ for $\psi_0 = T = 0$. The substitution of this into Eq. (12) leads to the system

$$\left(E - \omega\right)\psi_{A,B}^{(0)} \pm i\left(\gamma_0 - \gamma_2\left|\psi_{A,B}^{(0)}\right|^2\right)\psi_{A,B}^{(0)} = 0, \quad (14)$$

which gives rise to a continuous family of symmetric nonlinear Fano resonances, with $\omega = E$ and

$$\psi_{A,B}^{(0)} = i\sqrt{\gamma_0/\gamma_2}\exp\left(\pm i\delta\right), \quad (15)$$

$$\cos\delta = (2V)^{-1}\sqrt{\left(4C^2 - E^2\right)\left(\gamma_2/\gamma_0\right)P_{\text{in}}} \quad (16)$$

(recall the above EIT-like symmetric family, corresponding to $T \equiv 1$, had $\phi \neq \sqrt{\gamma_0/\gamma_2}$ and $\cos\delta = 0$). Equation (16) imposes condition $\cos^2\delta \leq 1$, i.e.,

$$P_{\text{in}} \leq 4V^2\left(\gamma_0/\gamma_2\right)\left(4C^2 - E^2\right)^{-1}. \quad (17)$$

Thus, at $\omega = E$, the continuous family of the symmetric nonlinear Fano resonances exists in this interval of the intensity of the incident wave. The novelty of the result is that the Fano resonance is usually obtained as an isolated solution.

The asymmetric scattering regimes. Equation (14) for the nonlinear Fano resonances also admits two *ultimate asymmetric states*, with the vanishing excitation at one of the $\mathcal{PT}$ elements: $\omega = E$ and

$$\psi_A^{(0)} = \sqrt{\gamma_0/\gamma_2}, \psi_B^{(0)} = 0, \quad (18)$$
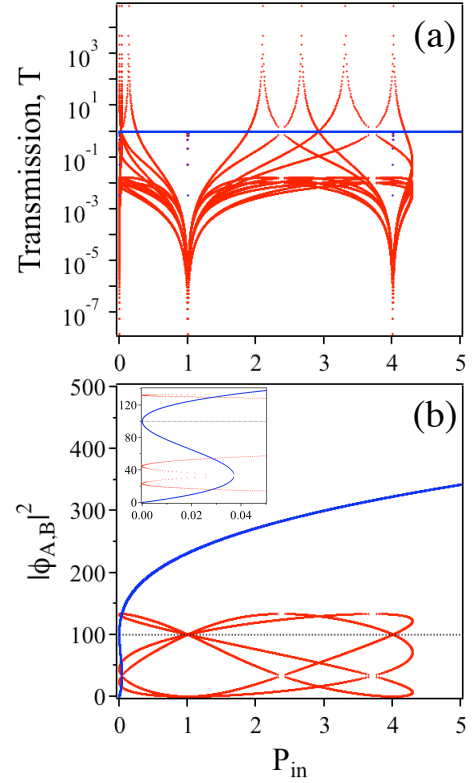

FIG. 3. (Color online) The transmission coefficient (a) and excitation intensity of the side-coupled $\mathcal{PT}$ elements (b) for $\gamma_0 = 0.01$, $\gamma_2 = 0.0001$, $V = 0.2$, $C = 1$, and $\omega = E = 0.1$. The nonlinear Fano resonances correspond to $T = 0$. The red and blue curves depict, respectively, the full multitude of asymmetric scattering regimes, produced by Eq. (12), and the symmetric one corresponding to Eq. (13). The inset in (b) zooms the tristability region in the latter case. The horizontal dotted line corresponds to Eq. (15).

or vice versa, with $A \rightleftarrows B$. In either case, this solution exists at $P_{\text{in}} = V^2\left(\gamma_0/\gamma_2\right)\left(4C^2 - E^2\right)^{-1}$. This point falls into the range (17) of the existence of the symmetric Fano-resonance solutions, which implies intrinsic bistability of the nonlinear Fano resonances.

Equation (12) gives rise to other asymmetric scattering regimes, which, in particular, may produce a strong resonant amplification of the transmitted wave. The complete set of the asymmetric scattering states is depicted by the red curves in Fig. 3.

Stability. The stability of the above analytical solutions was checked in direct simulations of Eqs. (7)-(9). The results demonstrate that the ultimate asymmetric state (18) with the nonzero excitation at the linear-loss element is stable, while its counterpart with the excitation at the linear-gain element is unstable, transforming itself into an oscillatory mode (apparently, a limit cycle), see left panels in Fig. 4. These results can be readily understood following the similarity to previously studied systems composed of coupled cores with the linear gain
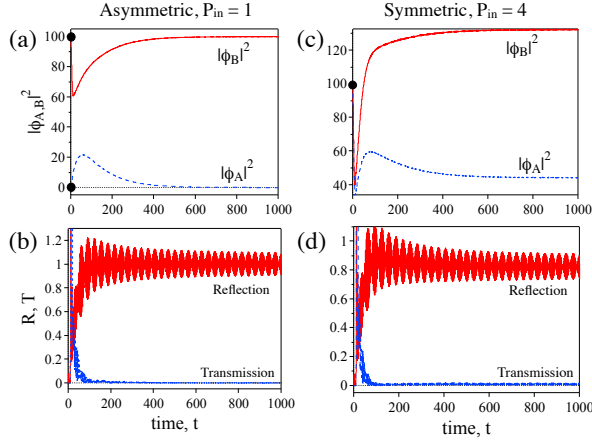
FIG. 4. (Color online) The perturbed evolution of those asymmetric (left) and symmetric (right) Fano-resonance modes from Fig. 3 which are dynamically unstable. Filled circles indicate the initial states.

and loss acting separately in them, which also give rise to a pair of stable and unstable modes [12].

Symmetric Fano-resonance modes (15) are unstable, also developing intrinsic oscillations, with a very low transmissivity and spontaneously broken symmetry between the $\mathcal{PT}$ elements, $|\psi_A| \neq |\psi_B|$, as shown in the right panels in Fig. 4. In fact, such asymmetric states correspond to the nearly perfect Fano resonance in Fig. 3 at $P_{\text{in}} = 4$. The resonantly-amplified transmission regimes are also unstable, due to enhancement of the field on the linear-gain element.

Nonpropagating modes. The dispersion relation for Eq. (8) demonstrates that frequencies of the propagating waves belong to the respective phonon band, $|\omega| < 2C$. Applying an excitation above the band, with $\omega > 2C$, it is possible to find the corresponding exact solutions for localized modes pinned to the $\mathcal{PT}$ complex:

$$\psi_n = \frac{V\left(\tilde{\psi}_A + \tilde{\psi}_B\right)}{2\omega - \sqrt{\omega^2 + 4C^2}} \left(\frac{\sqrt{\omega^2 + 4C^2} + \omega}{2C}\right)^{-|n|}, \quad (19)$$

where $\tilde{\psi}_A$ and $\tilde{\psi}_B$ are given by the above solutions for the symmetric and asymmetric modes, i.e., severally, Eqs. (2), (3) and (5), (6), with V and E replaced by $\tilde{V} \equiv V^2/\left(2\omega - \sqrt{\omega^2 + 4C^2}\right)$ and $\tilde{E} \equiv E + \tilde{V}$. This means that the solution for the nonpropagating symmetric modes remains explicit, while Eq. (5) for the asymmetric mode takes the form of a quartic equation for ω_{AS}.

In conclusion, we have introduced the species of $\mathcal{PT}$-symmetric systems with the balanced nonlinear gain and loss. For the dimer system, we have produced a complete set of analytical solutions, which feature the spontaneous symmetry breaking and multistability, not achievable in previously studied $\mathcal{PT}$-symmetric systems. We have demonstrated, also in the analytical form, that the symmetric and asymmetric excitations in the dimer, if it is coupled to the linear chain, give rise to a variety of novel nonlinear Fano resonances, including the bistability between them, and simultaneously to perfect-transmission regimes resembling EIT, as well as to the resonantly-amplified transmission. The coexistence of these scattering channels suggests an application to the design of data-processing schemes. Nonpropagating modes in the chain, pinned to the $\mathcal{PT}$ scatterer, were found too.

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